Filtration, automorphisms and classification of the infinite dimensional odd Contact superalgebras

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Abstract: The principal filtration of the infinite-dimensional odd Contact Lie superalgebra over a field of characteristic $p > 2$ is proved to be invariant under the automorphism group by investigating ad-nilpotent elements and determining certain invariants such as subalgebras generated by some ad-nilpotent elements. Then, it is proved that two automorphisms coincide if and only if they coincide on the -1 component with respect to the principal grading. Finally, all the odd Contact superalgebras are classified up to isomorphisms.

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0. Introduction

As is well known, filtration techniques are of great importance in the structure and classification theories of Lie (super)algebras. A descending filtration of a Lie superalgebra L is a sequence of $\mathbb{Z}_2$-graded spaces $L = L_0 \supset L_1 \supset \cdots$ for which $[L_i, L_j] \subset L_{i+j}$ holds for all i, j. In the situation that the Killing form on a simple Lie (super)algebra is degenerate, the filtration structure plays a particular role. A filtration $L = L_0 \supset L_1 \supset \cdots$ of a Lie superalgebra L is said to be invariant provided that $\varphi(L_i) \subset L_i$ for all i and all automorphisms φ of L. We know that the simple Lie (super)algebras of Cartan type possess various natural filtration structures, for which the invariance may be used to make an insight for the intrinsic properties and the automorphism groups of those Lie (super)algebras. The filtration structures have been studied for the finite dimensional Lie superalgebras of Cartan type, for example, in [4, 7, 9] the invariance of the natural filtrations was determined for the generalized Witt superalgebras, the special superalgebras, the Hamiltonian superalgbras and the odd Hamiltonian superalgebras.

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The reader is also refereed to [5, 8] for the similar work on certain infinite dimensional modular Lie superalgebras of Cartan type. Let us state the main results of this paper. Write $KO(n, n+1)$ for the infinite dimensional odd Contact Lie superalgebras over a field of prime characteristic (see Sec.1 for a definition and more details). $KO(n, n+1)$ has a canonical filtration structure known as principal. By means of characterizing ad-nilpotent elements of $KO(n, n+1)$ we first obtain in this paper that:

- (Theorem 4.1) The principal filtration of the odd Contact Lie superalgebra is invariant under the automorphism group of the Lie superalgebra.

As a consequence the automorphisms can be characterized as follows:

- (Theorem 4.4) Two automorphisms of the odd Contact Lie superalgebra coincide if and only if they coincide on the -1 component with respect to the principal grading.

Finally we classify all the infinite dimensional odd Contact Lie superalgebras up to isomorphisms:

- (Theorem 4.5) $KO(n, n+1) \cong KO(m, m+1)$ if and only if $n = m$.

1. Preliminaries

Throughout $\mathbb{F}$ is a field of characteristic $p > 2$; $\mathbb{Z}_2 := \{\bar{0}, \bar{1}\}$ is the additive group of two elements; $\mathbb{N}$ and $\mathbb{N}_0$ are the sets of positive integers and nonnegative integers, respectively. Let $\mathcal{O}(n)$ be the divided power algebra with $\mathbb{F}$-basis $\{x^{(\alpha)} \mid \alpha \in \mathbb{N}_0^n\}$. Note that $x^{(0)} := 1 \in \mathcal{O}(n)$, where $0 = (0, \ldots, 0) \in \mathbb{N}_0^n$. For $\varepsilon_i := (\delta_{i1}, \delta_{i2}, \ldots, \delta_{in}) \in \mathbb{N}_0^n$, write x_i for $x^{(\varepsilon_i)}$, where $i = 1, \ldots, n$. Let $\Lambda(m)$ be the exterior superalgebra over $\mathbb{F}$ in m variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$. Set

$$\mathbb{B}(m) := \left\{\langle i_1, i_2, \ldots, i_k\rangle \mid n+1 \leq i_1 < i_2 < \cdots < i_k \leq n+m,\ k \in \overline{0, m}\right\}.$$

For $u := \langle i_1, i_2, \ldots, i_k\rangle \in \mathbb{B}(m)$, write $|u| := k$ and $x^u := x_{i_1} x_{i_2} \cdots x_{i_k}$. Notice that we also denote the index set $\{i_1, i_2, \ldots, i_k\}$ by u itself. For $u, v \in \mathbb{B}(m)$ with $u \cap v = \emptyset$, define $u + v$ to be the uniquely determined element $w \in \mathbb{B}(m)$ such that $w = u \cup v$. If $v \subset u$, define $u - v$ to be $w \in \mathbb{B}(m)$ such that $w = u \setminus v$. Clearly, the associative superalgebra $\mathcal{O}(n, m) := \mathcal{O}(n) \otimes \Lambda(m)$ has a so-called *standard* $\mathbb{F}$-*basis* $\{x^{(\alpha)} x^u \mid (\alpha, u) \in \mathbb{N}_0^n \times \mathbb{B}(m)\}$.

Let ∂_r be the superderivations of $\mathcal{O}(n, m)$ defined by $\partial_r(x^{(\alpha)}) = x^{(\alpha - \varepsilon_r)}$ for $r \in \overline{1, n}$ and $\partial_r(x_s) = \delta_{rs}$ for $r, s \in \overline{1, n+m}$. Here $\overline{1, n}$ is the set of integers $1, 2, \ldots, n$. The generalized Witt superalgebra $W(n, m)$ is $\mathbb{F}$-spanned by all $f_r \partial_r$, where $f_r \in \mathcal{O}(n, m)$, $r \in \overline{1, n+m}$. Note that $W(n, m)$ is a free $\mathcal{O}(n, m)$-module with basis $\{\partial_r \mid r \in \overline{1, n+m}\}$. In particular, $W(n, m)$ has a *standard* $\mathbb{F}$-*basis* $\{x^{(\alpha)} x^u \partial_r \mid (\alpha, u, r) \in \mathbb{N}_0^n \times \mathbb{B}(m) \times \overline{1, n+m}\}$.

For an n-tuple $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, put $|\alpha| := \sum_{i=1}^n \alpha_i$. When $m = n+1$, we usually write $\mathcal{O} := \mathcal{O}(n, n+1)$, $W := W(n, n+1)$, $\mathbb{A} := \mathbb{N}_0^n$ and $\mathbb{B} := \mathbb{B}(n+1)$. If $u := \langle i_1, \ldots, i_r\rangle \in \mathbb{B}$, put

$$\| u \| := \begin{cases} |u| + 1, & \text{if } 2n+1 \in u \\ |u|, & \text{if } 2n+1 \notin u. \end{cases}$$

Recall the standard $\mathbb{Z}$-grading, $\mathcal{O} = \oplus_{i \geq 0} \mathcal{O}_{\mathbf{s},[i]}$, where

$$\mathcal{O}_{\mathbf{s},[i]} := \mathrm{span}_{\mathbb{F}}\{x^{(\alpha)} x^u \mid |\alpha| + |u| = i, \alpha \in \mathbb{A}, u \in \mathbb{B}\}.$$

It induces naturally the standard grading $W = \oplus_{i\geq -1} W_{\mathbf{s},[i]}$, where

$$W_{\mathbf{s},[i]} := \mathrm{span}_{\mathbb{F}}\{f\partial_j \mid f \in \mathcal{O}_{\mathbf{s},[i+1]}, j \in \overline{1, 2n+1}\}.$$

The standard gradings of $\mathcal{O}$ and W are of type $(1,\ldots,1 \mid 1,\ldots,1)$. Let $W_{\mathbf{s},i} = \sum_{j\geq i} W_{\mathbf{s},[j]}$. Then $(W_{\mathbf{s},i})_{i\geq -1}$ is called the standard filtration of W.

We shall also use the principal grading $\mathcal{O} = \oplus_{i\geq 0}\mathcal{O}_{\mathbf{p},[i]}$, where

$$\mathcal{O}_{\mathbf{p},[i]} := \mathrm{span}_{\mathbb{F}}\{x^{(\alpha)}x^u \mid |\alpha| + \|u\| = i,\ \alpha \in \mathbb{A},\ u \in \mathbb{B}\},$$

and the principal grading $W = \oplus_{i\geq -2} W_{\mathbf{p},[i]}$, where

$$W_{\mathbf{p},[i]} := \mathrm{span}_{\mathbb{F}}\{f\partial_j \mid f \in \mathcal{O}_{\mathbf{p},[i+1+\delta_{j,2n+1}]}, j \in \overline{1, 2n+1}\}$$

(cf. [1]). The principal gradings of $\mathcal{O}$ and W are of type $(1,\ldots,1 \mid 1,\ldots,1,2)$.

For a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$, we write $\mathrm{p}(x) := \theta$ for the parity of a $\mathbb{Z}_2$-homogeneous element $x \in V_\theta$, $\theta \in \mathbb{Z}_2$. Once the symbol $\mathrm{p}(x)$ appears, it will imply that x is a $\mathbb{Z}_2$-homogeneous element.

The odd Contact superalgebra, which is a subalgebra of W, is defined as follows (see [1] for more details):

$$KO(n, n+1) := \{D_{KO}(a) \mid a \in \mathcal{O}\},$$

where

$$D_{KO}(a) := T_H(a) + (-1)^{\mathrm{p}(a)}\partial_{2n+1}(a)E + (E(a) - 2a)\partial_{2n+1},$$

$$E := \sum_{i=1}^{2n} x_i\partial_i, \quad T_H(a) := \sum_{i=1}^{2n}(-1)^{\mu(i')\mathrm{p}(a)}\partial_{i'}(a)\partial_i,$$

$$i' := \begin{cases} i+n, & \text{if } i \in \overline{1,n} \\ i-n, & \text{if } i \in \overline{n+1, 2n}, \end{cases} \qquad \mu(i) := \begin{cases} \bar{0}, & \text{if } i \in \overline{1,n} \\ \bar{1}, & \text{if } i \in \overline{n+1, 2n+1}. \end{cases}$$

For the operator T_H and further information, the reader is referred to [3]. Note that for $a, b \in \mathcal{O}$ (see [1]),

$$[D_{KO}(a), D_{KO}(b)] = D_{KO}(D_{KO}(a)(b) - (-1)^{\mathrm{p}(a)}2\partial_{2n+1}(a)b). \tag{1.1}$$

For simplicity, we usually write KO for $KO(n, n+1)$. Note that KO has a so-called principal $\mathbb{Z}$-grading structure denoted by

$$KO = \oplus_{i\geq -2} KO_{\mathbf{p},[i]}, \quad \text{where } KO_{\mathbf{p},[i]} := KO \cap W_{\mathbf{p},[i]}.$$

In particular,

$$\begin{aligned} &KO_{\mathbf{p},[-2]} = \mathbb{F} \cdot D_{KO}(1), \\ &KO_{\mathbf{p},[-1]} = \mathrm{span}_{\mathbb{F}}\{D_{KO}(x_i) \mid i \in \overline{1, 2n}\}, \\ &KO_{\mathbf{p},[0]} = \mathrm{span}_{\mathbb{F}}\{D_{KO}(x_{2n+1}), D_{KO}(x_ix_j) \mid 1 \leq i \leq j \leq 2n\}. \end{aligned} \tag{1.2}$$

Let

$$X_{\mathbf{p},i} := \sum_{j\geq i} X_{\mathbf{p},[j]}, \quad \text{where } X = W \text{ or } KO.$$

Then $(X_{\mathbf{p},i})_{i\geq -2}$ is called the principal filtration of X. Recall that the infinite-dimensional generalized Witt superalgebra $W(n,n)$ contains the following Lie superalgebra as a subalgebra (see [1, 2]):

$$SHO'(n,n) := \{T_H(a) \mid a \in \mathcal{O}(n,n), \Delta(a) = 0\}, \quad \text{where } \Delta := \textstyle\sum_{i=1}^n \partial_i\partial_{i'}.$$

Convention: In the sequel we shall write $KO_{[i]}$ and KO_i for $KO_{\mathbf{p},[i]}$ and $KO_{\mathbf{p},i}$, respectively.

2. Ad-nilpotent elements

Let L be a Lie superalgebra. An element $y \in L$ is ad-nilpotent if as a transformation ad$y$ is nilpotent on L. Let R be a subalgebra of L. Put

$$\begin{aligned}
\mathrm{nil}(R) &:= \text{the set of the elements in } R \text{ which are ad-nilpotent on } L,\\
\mathrm{span}_{\mathbb{F}}\mathrm{nil}(R) &:= \text{the subspace spanned by } \mathrm{nil}(R),\\
\mathrm{Nil}(R) &:= \text{the subalgebra of } L \text{ generated by } \mathrm{nil}(R).
\end{aligned}$$

Lemma 2.1. *Suppose $a \in \mathcal{O}$ is of $\mathbb{Z}$-degree 2 with respect to the standard grading. If $\partial_{2n+1}(a) = 0$, that is, if $a \in \mathcal{O}(n,n)$, then*

$$[D_{KO}(a), D_{KO}(b)] = D_{KO}(T_H(a)(b)) \quad \textit{for all } b \in \mathcal{O}.$$

Proof. It follows from (1.1). □

A nonempty subset S of a Lie superalgebra L is called a Lie-super subset if S is closed under the multiplication of L and it spans a sub-superspace (and then is a Lie superalgebra). A slight modification of [6, Theorem 1.3.1] yields the following lemma.

Lemma 2.2. *Suppose V is a vector superspace over $\mathbb{F}$ and S a Lie-super subset of Lie superalgebra $\mathfrak{gl}(V)$. If S consists of nilpotent linear transformations* $\mathrm{span}_{\mathbb{F}}S$ *is of finite dimension, then* $\mathrm{span}_{\mathbb{F}}S$ *is strictly triangulable on V, that is, there is a finite sequence $(V_i)_{0\leq i\leq m}$ of sub-superspaces such that*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_m = V; \quad x(V_i) \subset V_{i-1} \quad \textit{for all } x \in \mathrm{span}_{\mathbb{F}}S.$$

Lemma 2.3. $W_{\mathbf{p},1} \subseteq \mathrm{nil}(W)$.

Proof. There exists a sub-superspace $V \subseteq W_{\mathbf{s},1}$ such that

$$W_{\mathbf{p},1} = \mathrm{span}_{\mathbb{F}}\{x_{2n+1}\partial_i \mid i \in \overline{1,2n}\} + V.$$

For $E \in W_{\mathbf{p},1}$, write $E = E_0 + E_1$, where $E_0 \in \mathrm{span}_{\mathbb{F}}\{x_{2n+1}\partial_i \mid i \in \overline{1,2n}\}$, $E_1 \in V$. Put $E_2 := [E_0, E_1]$. Then there exists an n-tuple $\underline{t}$ of positive integers such that $E_1, E_2 \in W_{\mathbf{s},1}(n, n+1; \underline{t}) := W(n, n+1; \underline{t}) \cap W_{\mathbf{s},1}$ [see [3] for a definition of $W(n, n+1; \underline{t})$]. Note that

$$S := \mathrm{span}_{\mathbb{F}}\{\mathrm{ad}(x_{2n+1}\partial_i) \mid i \in \overline{1,2n}\} \cup \mathrm{ad}W_{\mathbf{s},1}(n, n+1; \underline{t})$$

is a Lie-super subset of the general linear Lie superalgebra $\mathfrak{gl}(W)$. A direct computation shows that $\mathrm{span}_{\mathbb{F}}\{x_{2n+1}\partial_i \mid i \in \overline{1,2n}\} \subseteq \mathrm{nil}(W)$. By virtue of [8, Theorem 2.5], we have $W_{\mathbf{s},1}(n, n+1; \underline{t}) \subseteq \mathrm{nil}(W)$. Then Lemma 2.2 ensures that $E \in \mathrm{nil}(W)$. □

Lemma 2.4. (1) $KO_1 \subseteq \mathrm{nil}(KO)$.

(2) *Suppose $y = y_{[i]} + y_{i+1} \in \mathrm{nil}(KO_i)$, where $y_{[i]} \in KO_{[i]}$ and $y_{i+1} \in KO_{i+1}$. Then $y_{[i]} \in \mathrm{nil}(KO_{[i]})$.*

(3) *Suppose $y = y_{[-1]} + y_0 \in \mathrm{nil}(KO_{\bar{0}})$, where $y_{[-1]} \in KO_{[-1]} \cap KO_{\bar{0}}$ and $y_0 \in KO_0 \cap KO_{\bar{0}}$. Then $y_{[-1]} = 0$.*

(4) $\mathrm{Nil}(KO_{\bar{0}}) = \mathrm{Nil}(KO_{[0]} \cap KO_{\bar{0}}) + KO_1 \cap KO_{\bar{0}}$.

Proof. (1) Since $KO_1 \subseteq W_{\mathbf{p},1}$, by virtue of Lemma 2.3 we get $KO_1 \subseteq \mathrm{nil}(KO)$.

(2) Similar to [8, Lemma 2.7], one may prove (2).

(3) Suppose $y_{[-1]} = \sum_{i=1}^{n} a_i D_{KO}(x_{i'})$, where $a_i \in \mathbb{F}$. Note that $D_{KO}(x^{((k+1)\varepsilon_j)}) \in KO$ for $k \in \mathbb{N}$. If $a_j \neq 0$ for some $j \in \overline{1,n}$, then $y_{[-1]}$ is not ad-nilpotent, since $(\mathrm{ad}y_{[-1]})^k(D_{KO}(x^{((k+1)\varepsilon_j)})) = a_j^k D_{KO}(x_j) \neq 0$. This contradicts (2). Therefore, $a_j = 0$ for all $j \in \overline{1,n}$, that is, $y_{[-1]} = 0$.

(4) It follows from (1) and (3). $\square$

Lemma 2.5. *If $i \neq j' \in \overline{1,2n}$, then $(T_H(x_i x_j))^{2p} = 0$.*

Proof. Note that

$$T_H(x_i x_j) = (-1)^{\mu(i)+\mu(i)\mu(j)} x_j \partial_{i'} + (-1)^{\mu(j)} x_i \partial_{j'}, \tag{2.1}$$

$(x_j\partial_{i'})^p = (x_i\partial_{j'})^p = 0$ and $[x_j\partial_{i'}, x_i\partial_{j'}] = 0$ for all $i \neq j' \in \overline{1,2n}$. In combination with (2.1), we have $(T_H(x_i x_j))^{2p} = 0$. $\square$

Lemma 2.6. *For $i, j \in \overline{1,2n}$, the following statements hold.*

(1) *If $f \in \mathcal{O}$, $D_{KO}(f) \in KO_{[0]}$ and $\partial_{2n+1}(f) \neq 0$, then $D_{KO}(f) \notin \mathrm{nil}(KO_{[0]})$.*

(2) *Suppose $i \neq j'$ and $a_i \in \mathbb{F}$ for all $i \in \overline{1,n}$. Then*

$$\begin{aligned}
&D_{KO}(x_i x_j) \in \mathrm{nil}(KO_{[0]}), && \sum_{i=1}^{n} a_i D_{KO}(x_i x_{i'}) \notin \mathrm{nil}(KO_{[0]}) \textit{ or is } 0,\\
&D_{KO}(x_i x_{i'}) \notin \mathrm{Nil}(KO_{[0]}), && D_{KO}(x_{2n+1}) \notin \mathrm{Nil}(KO_{[0]}),\\
&D_{KO}(x_i x_{i'} - x_j x_{j'}) \in \mathrm{Nil}(KO_{[0]}).
\end{aligned}$$

Proof. (1) Suppose $f \in \mathcal{O}$, $D_{KO}(f) \in KO_{[0]}$ and $\partial_{2n+1}(f) \neq 0$. Then there exist $0 \neq a \in \mathbb{F}$ and $f_0 \in \mathcal{O}(n,n)$ such that $f = ax_{2n+1} + f_0$. Since

$$[D_{KO}(ax_{2n+1} + f_0), D_{KO}(1)] = 2aD_{KO}(1),$$

we have $D_{KO}(f) \notin \mathrm{nil}(KO_{[0]})$.

(2) Applying Lemma 2.1 we obtain by induction on k that

$$(\mathrm{ad}D_{KO}(x_i x_j))^k(D_{KO}(f)) = D_{KO}(T_H(x_i x_j)^k(f)) \text{ for all } k \in \mathbb{N},$$

where $f \in \mathcal{O}$. Since $\mathrm{Ker}(D_{KO}) = 0$, one sees that $D_{KO}(x_i x_j)$ is ad-nilpotent if and only if $T_H(x_i x_j)$ is a nilpotent transformation of $\mathcal{O}$. Then by Lemma 2.5, we know that

$$D_{KO}(x_i x_j) \in \mathrm{nil}(KO_{[0]}) \quad \text{for all } i \neq j' \in \overline{1,2n}.$$

Note that

$$\Big[\sum_{i=1}^{n} a_i D_{KO}(x_i x_{i'}), D_{KO}(x_j x_{2n+1})\Big] = -a_j D_{KO}(x_j x_{2n+1}),\ j \in \overline{1,n}.$$

If $\sum_{i=1}^{n} a_i D_{KO}(x_i x_{i'}) \neq 0$ then

$$\sum_{i=1}^{n} a_i D_{KO}(x_i x_{i'}) \notin \mathrm{nil}(KO_{[0]}).$$

It follows from (1) that $\mathrm{nil}(KO_{[0]}) \subseteq SHO'(n,n)$. Therefore, $\mathrm{Nil}(KO_{[0]}) \subseteq SHO'(n,n)$. But $D_{KO}(x_i x_{i'})$, $D_{KO}(x_{2n+1}) \notin SHO'(n,n)$, thus

$$D_{KO}(x_i x_{i'}), D_{KO}(x_{2n+1}) \notin \mathrm{Nil}(KO_{[0]}).$$

By virtue of the fact that

$$[D_{KO}(x_i x_j), D_{KO}(x_{i'} x_{j'})] = -D_{KO}(x_i x_{i'} - x_j x_{j'}),$$

we have $D_{KO}(x_i x_{i'} - x_j x_{j'}) \in \mathrm{Nil}(KO_{[0]})$. □

3. Invariant subalgebras

Let

$$\begin{aligned}
&\mathfrak{T} := \mathrm{Nor}_{KO_{\bar{0}}}(\mathrm{Nil}(KO_{\bar{0}})),\\
&\mathfrak{Q} := \{y \in KO_{\bar{1}} \mid [y, KO_{\bar{1}}] \subseteq T\},\\
&\mathfrak{M} := \{y \in KO_{\bar{1}} \mid [y, \mathfrak{Q}] \subseteq \mathrm{Nil}(KO_{\bar{0}})\}.
\end{aligned}$$

It is easy to see that $\mathfrak{T}$ is an invariant subspace under the automorphisms of KO and so are $\mathfrak{Q}$ and $\mathfrak{M}$.

Proposition 3.1. *$\mathfrak{T} = KO_0 \cap KO_{\bar{0}}$. In particular, $KO_0 \cap KO_{\bar{0}}$ is an invariant subalgebra of KO.*

Proof. Let

$$y = \sum_{i=1}^{n} a_i D_{KO}(x_{i'}) + y'' \in \mathfrak{T},$$

where $a_i \in \mathbb{F}$ for all $i \in \overline{1,n}$, $y'' \in KO_0 \cap KO_{\bar{0}}$. Assume that $a_j \neq 0$ for some $j \in \overline{1,n}$. Take $j \neq k \in \overline{1,n}$. By Lemma 2.4(4), we have $D_{KO}(x_j x_k x_{k'}) \in KO_1 \cap KO_{\bar{0}} \subseteq \mathrm{Nil}(KO_{\bar{0}})$. Then

$$\begin{aligned}
&-a_j D_{KO}(x_k x_{k'}) - a_k D_{KO}(x_j x_{k'}) + h\\
=\ &\Big[\sum_{i=1}^{n} a_i D_{KO}(x_{i'}) + y'', D_{KO}(x_j x_k x_{k'})\Big] \in \mathrm{Nil}(KO_{\bar{0}}),
\end{aligned}$$

where $h \in KO_1 \cap KO_{\bar{0}}$. This contradicts Lemma 2.6(2) and then $\mathfrak{T} \subseteq KO_0 \cap KO_{\bar{0}}$. On the other hand, by (1.2) and Lemma 2.6(2), we have

$$KO_0 \cap KO_{\bar{0}} = \mathrm{span}_{\mathbb{F}}\mathrm{nil}(KO_{\bar{0}}) + Y,$$

where $Y := \mathrm{span}_{\mathbb{F}}\{D_{KO}(x_i x_{i'}), D_{KO}(x_{2n+1}) \mid i \in \overline{1,n}\}$. By (1.1), we have

$$[Y, \mathrm{Nil}(KO_{\bar{0}})] \subseteq \mathrm{Nil}(KO_{\bar{0}}).$$

Then

$$[KO_0 \cap KO_{\bar{0}}, \mathrm{Nil}(KO_{\bar{0}})] = [\mathrm{span}_{\mathbb{F}}\mathrm{nil}(KO_{\bar{0}}) + Y, \mathrm{Nil}(KO_{\bar{0}})] \subseteq \mathrm{Nil}(KO_{\bar{0}}).$$

The proof is complete. □

Lemma 3.2. $\mathfrak{Q} \subseteq \mathrm{span}_{\mathbb{F}}\{D_{KO}(x_i x_j) \mid 1 \leq i \leq j \leq n\} + KO_1 \cap KO_{\bar{1}}$.

Proof. For $y \in \mathfrak{Q}$, we may write

$$y = D_{KO}(a) + y',$$

where $a \in \mathbb{F}$, $y' \in KO_{-1} \cap KO_{\bar{1}}$. Note that $D_{KO}(x_{1'} x_{2n+1}) \in KO_{\bar{1}}$. Then

$$-2aD_{KO}(x_{1'}) + h = [D_{KO}(a) + y', D_{KO}(x_{1'} x_{2n+1})] \in KO_0 \cap KO_{\bar{0}},$$

where $h \in KO_0 \cap KO_{\bar{0}}$. Then $a = 0$. Thus we may write

$$y = \sum_{i=1}^{n} a_i D_{KO}(x_i) + y'',$$

where $a_i \in \mathbb{F}$ for all $i \in \overline{1,n}$, $y'' \in KO_0 \cap KO_{\bar{1}}$. If $a_j \neq 0$ for some $j \in \overline{1,n}$, take $j \neq k \in \overline{1,n}$. Note that $D_{KO}(x_{j'} x_{k'}) \in KO_{\bar{1}}$. We have

$$D_{KO}(a_j x_{k'} - a_k x_{j'}) + h = \Big[\sum_{i=1}^{n} a_i D_{KO}(x_i) + y'', D_{KO}(x_{j'} x_{k'})\Big] \in KO_0 \cap KO_{\bar{0}},$$

where $h \in KO_0 \cap KO_{\bar{0}}$, contradicting that $a_j \neq 0$. Thus we may write

$$y = \sum_{1 \leq i \leq j \leq n} a_{ij} D_{KO}(x_i x_j) + \sum_{1 \leq i < j \leq n} b_{ij} D_{KO}(x_{i'} x_{j'}) + y'',$$

where $a_{ij}, b_{ij} \in \mathbb{F}$ for all $1 \leq i \leq j \leq n$, $y'' \in KO_1 \cap KO_{\bar{1}}$. Assume that $b_{kl} \neq 0$ for some $1 \leq k < l \leq n$. Since $D_{KO}(x_k) \in KO_{\bar{1}}$, we have

$$\sum_{i=1}^{k-1} -b_{ik} D_{KO}(x_{i'}) + \sum_{i=k+1}^{n} b_{ki} D_{KO}(x_{i'}) + h = [y, D_{KO}(x_k)] \in KO_0 \cap KO_{\bar{0}},$$

where $h \in KO_0 \cap KO_{\bar{0}}$, contradicting the assumption that $b_{kl} \neq 0$. Then

$$\mathfrak{Q} \subseteq \mathrm{span}_{\mathbb{F}}\{D_{KO}(x_i x_j) \mid 1 \leq i \leq j \leq n\} + KO_1 \cap KO_{\bar{1}}$$

and this completes the proof. □

Remark 3.3. *For $i \neq j \in \overline{1,n}$, we have $D_{KO}(x_i x_{i'} x_{2n+1}) \in \mathfrak{Q}, D_{KO}(x_{i'} x_j x_{j'}) \in \mathfrak{Q}$.*

Proposition 3.4. *$\mathfrak{M} = KO_0 \cap KO_{\bar{1}}$. In particular, $KO_0 \cap KO_{\bar{1}}$ is an invariant subalgebra of KO.*

Proof. By (1.2), Lemmas 2.4(4), 2.6(2) and 3.2, we have

$$\begin{aligned}[KO_0 \cap KO_{\bar{1}}, \mathfrak{Q}] &\subseteq [KO_0 \cap KO_{\bar{1}}, \mathrm{span}_{\mathbb{F}}\{D_{KO}(x_i x_j) \mid 1 \leq i \leq j \leq n\} + KO_1 \cap KO_{\bar{1}}] \\ &\subseteq \mathrm{Nil}(KO_{\bar{0}}).\end{aligned}$$

Hence $KO_0 \cap KO_{\bar{1}} \subseteq \mathfrak{M}$. Conversely, for $y \in \mathfrak{M}$, we may write

$$y = D_{KO}(a) + y',$$

where $a \in \mathbb{F}$, $y' \in KO_{-1} \cap KO_{\bar{1}}$. By Remark 3.3, we have

$$2aD_{KO}(x_i x_{i'}) + h = [D_{KO}(a) + y', D_{KO}(x_i x_{i'} x_{2n+1})] \in \mathrm{Nil}(KO_{\bar{0}}),$$

where $h \in KO_1 \cap KO_{\bar{0}}$. By Lemma 2.6(2), we have $a = 0$. Thus we may write

$$y = \sum_{i=1}^{n} a_i D_{KO}(x_i) + y'',$$

where $a_i \in \mathbb{F}$ for all $i \in \overline{1,n}$, $y'' \in KO_0 \cap KO_{\bar{1}}$. Assume that $a_j \neq 0$ for some $j \in \overline{1,n}$. Take $j \neq k \in \overline{1,n}$. By Remark 3.3, we have

$$\begin{aligned} & a_j D_{KO}(x_k x_{k'}) - a_k D_{KO}(x_{j'} x_k) + h \\ = & \Big[\sum_{i=1}^{n} a_i D_{KO}(x_i) + y'', D_{KO}(x_{j'} x_k x_{k'})\Big] \in \mathrm{Nil}(KO_{\bar{0}}), \end{aligned}$$

where $h \in KO_1 \cap KO_{\bar{0}}$. By Lemma 2.6(2), we have $a_j = 0$. This contradicts the assumption that $a_j \neq 0$. Thus $\mathfrak{M} \subseteq KO_0 \cap KO_{\bar{1}}$ and the proof is complete. □

The key in this paper is the following proposition.

Proposition 3.5. *KO_0 is an invariant subalgebra of KO.*

Proof. It follows from Propositions 3.1 and 3.4. □

4. Filtration, automorphisms and classification

One of the main results is as follows, which is a direct consequence of Proposition 3.5 and the following Lemmas 4.2 and 4.3.

Theorem 4.1. *The principal filtration of KO is invariant under the automorphisms of KO, that is, $\varphi(KO_i) = KO_i$ for all $i \geq -2$ and all automorphisms φ of KO.* □

Lemma 4.2. *KO_{-1}/KO_0 is the unique irreducible KO_0-submodule of KO/KO_0. In particular, KO_{-1} is an invariant subalgebra of KO.*

Proof. Clearly, KO_{-1}/KO_0 is an irreducible KO_0-submodule. To show the uniqueness, suppose M/KO_0 is a nonzero KO_0-submodule of KO/KO_0, where $M \supset KO_0$ is a KO_0-submodule of KO. For $0 \neq y \in M$, one may write $y = D_{KO}(1) + y'$, where $y' \in KO_{-1}$. Then

$$[D_{KO}(x_i x_{2n+1}), D_{KO}(1) + y'] \in M \quad \text{for all } i \in \overline{1,2n}.$$

Note that $[D_{KO}(x_i x_{2n+1}), y'] \in M$. We have

$$2D_{KO}(x_i) = [D_{KO}(x_i x_{2n+1}), D_{KO}(1)] \in M$$

for all $i \in \overline{1,2n}$. Therefore, $KO_{-1}/KO_0 \subseteq M/KO_0$. The proof is complete. □

Lemma 4.3. *$KO_i = \{y \in KO_{i-1} \mid [y, KO_{-1}] \subseteq KO_{i-1}\}$ for all $i \geq 1$.*

Proof. Put
$$\mathfrak{h}_i := \{y \in KO_{i-1} \mid [y, KO_{-1}] \subseteq KO_{i-1}\}.$$
Clearly, $KO_i \subseteq \mathfrak{h}_i$. For all $y \in \mathfrak{h}_i$, we may write $y := \sum_{j\geq i-1} y_j$, where $y_j \in KO_{[j]}$. By the definition of $\mathfrak{h}_i$, $[y_{i-1}, KO_{-1}] = 0$. Let $y_{i-1} := \sum_{\alpha,u} b_{\alpha,u} D_{KO}(x^{(\alpha)}x^u)$, where $\alpha \in \mathbb{A}$, $u \in \mathbb{B}$, $D_{KO}(x^{(\alpha)}x^u) \in KO_{[i-1]} \subseteq KO_0$, $b_{\alpha,u} \in \mathbb{F}$. For any fixed $\beta \neq 0$ with $\beta_k \geq 1$ for some $k \in \overline{1,n}$, we have

$$\begin{aligned} 0 &= \Big[\sum_{\alpha,u} b_{\alpha,u} D_{KO}(x^{(\alpha)}x^u), D_{KO}(x_{k'})\Big] \\ &= \sum_{\alpha,u} b_{\alpha,u} D_{KO}(\partial_k(x^{(\alpha)}x^u) - (-1)^{\mathrm{p}(x^u)}\partial_{2n+1}(x^{(\alpha)}x^u)x_{k'}). \end{aligned}$$

Consequently, $b_{\alpha,u} = 0$ whenever $\alpha \neq 0$. It remains to consider the case $\alpha = 0$. Fix any $v \neq \emptyset$. If there is $l \in \overline{n+1, 2n}$ such that $l \in v$. Then

$$\begin{aligned} 0 &= \Big[\sum_{0,u} b_{0,u} D_{KO}(x^u), D_{KO}(x_{l'})\Big] \\ &= \sum_{0,u} b_{0,u} D_{KO}((-1)^{\mathrm{p}(x^u)}\partial_l(x^u) - (-1)^{\mathrm{p}(x^u)}\partial_{2n+1}(x^u)x_{l'}). \end{aligned}$$

It follows that $b_{0,v} = 0$, where $\langle 2n+1\rangle \neq v \in \mathbb{B}$. Taking $i \in \overline{1,2n}$, since
$$0 = [b_{0,\langle 2n+1\rangle} D_{KO}(x_{2n+1}), D_{KO}(x_i)] = b_{0,\langle 2n+1\rangle} D_{KO}(x_i),$$
we have $b_{0,\langle 2n+1\rangle} = 0$. Therefore, $y_{i-1} = 0$ and then $\mathfrak{h}_i \subset KO_i$. □

As a corollary we give a characterization of the automorphisms of KO:

Theorem 4.4. *Two automorphisms of KO coincide if and only if they coincide on the -1 component $KO_{[-1]}$.*

Proof. Let ϕ and ψ be the automorphisms of KO. It suffices to prove that $\phi\mid_{KO_{[-1]}} = \psi\mid_{KO_{[-1]}}$ implies $\phi = \psi$. As in [5, Corollary 18], using Theorem 4.1 one can prove that $\phi\mid_{KO_{-1}} = \psi\mid_{KO_{-1}}$. Since
$$\phi(D_{KO}(1)) = \phi([D_{KO}(x_1), D_{KO}(x_{1'})]) = \psi([D_{KO}(x_1), D_{KO}(x_{1'})]) = \psi(D_{KO}(1)),$$
we have $\phi\mid_{KO_{[-2]}} = \psi\mid_{KO_{[-2]}}$. Then $\phi = \psi$ and the proof is complete. □

We are now in position to state the final main result in this paper, which says that the parameter n defining the odd Contact superalgebra $KO(n, n+1)$ is intrinsic and then all the infinite-dimensional odd Contact superalgebras are classified up to isomorphisms.

Theorem 4.5. $KO(n, n+1) \cong KO(m, m+1)$ *if and only if* $n = m$.

Proof of Theorem 4.5. One direction is obvious. Assume that $\sigma : KO(n, n+1) \longrightarrow KO(m, m+1)$ is an isomorphism of Lie superalgebras. Clearly,
$$\sigma(\mathrm{Nor}_{KO(n,n+1)_{\bar 0}}(\mathrm{Nil}(KO(n,n+1)_{\bar 0}))) = \mathrm{Nor}_{KO(m,m+1)_{\bar 0}}(\mathrm{Nil}(KO(m,m+1)_{\bar 0})).$$
In view of the proof of Proposition 3.1, we have
$$KO(n, n+1)_0 \cap KO(n, n+1)_{\bar 0} = \mathrm{Nor}_{KO(n,n+1)_{\bar 0}}(\mathrm{Nil}(KO(n,n+1)_{\bar 0})).$$

Therefore,

$$\sigma(KO(n,n+1)_0 \cap KO(n,n+1)_{\bar{0}}) = KO(m,m+1)_0 \cap KO(m,m+1)_{\bar{0}}. \qquad (4.1)$$

Recall that

$$\mathfrak{Q} = \{y \in KO(n,n+1)_{\bar{1}} \mid [y, KO(n,n+1)_{\bar{1}}] \subseteq KO(n,n+1)_0 \cap KO(n,n+1)_{\bar{0}}\}$$

and

$$\mathfrak{M} = \{y \in KO(n,n+1)_{\bar{1}} \mid [y, \mathfrak{Q}] \subseteq \mathrm{Nil}(KO(n,n+1)_{\bar{0}})\}.$$

By Lemma 3.2 and Proposition 3.4, we have

$$\sigma(KO(n,n+1)_0 \cap KO(n,n+1)_{\bar{1}}) = KO(m,m+1)_0 \cap KO(m,m+1)_{\bar{1}}. \qquad (4.2)$$

By (4.1) and (4.2), we have

$$\sigma(KO(n,n+1)_0) = KO(m,m+1)_0.$$

Therefore σ induces an isomorphism of $\mathbb{Z}_2$-graded vector spaces

$$\sigma' : KO(n,n+1)/KO(n,n+1)_0 \longrightarrow KO(m,m+1)/KO(m,m+1)_0.$$

Since $KO(n,n+1)/KO(n,n+1)_0 \cong KO(n,n+1)_{[-2]} \oplus KO(n,n+1)_{[-1]}$ as $\mathbb{Z}_2$-graded vector spaces, we have

$$\dim(KO(n,n+1)_{[-2]} \oplus KO(n,n+1)_{[-1]}) = \dim(KO(m,m+1)_{[-2]} \oplus KO(m,m+1)_{[-1]}).$$

This implies that $n+1 = m+1$, that is, $n = m$. The proof is complete.

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